Nonlinear Stochastic Resonance with subthreshold rectangular pulses

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Abstract

We analyze the phenomenon of nonlinear stochastic resonance (SR) in noisy bistable systems driven by pulsed time periodic forces. The driving force contains, within each period, two pulses of equal constant amplitude and duration but opposite signs. Each pulse starts every half-period and its duration is varied. For *subthreshold* amplitudes, we study the dependence of the output signal-to-noise ratio (SNR) and the SR gain on the noise strength and the relative duration of the pulses. We find that the SR gains can reach values larger than unity, with maximum values showing a nonmonotonic dependence on the duration of the pulses.

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In recent work [1, 2], we have carried out a detailed analytical and numerical study of the nonlinear response of a noisy bistable system subject to a subthreshold time periodic driving force. The force considered in those works is such that it remains constant within the duration of each half period while switching its sign every half period. We have focused our attention on the analysis of the dependence of the output signal-to-noise ratio and the corresponding SR gain on the noise strength. The analytical study was based on a two-state approximation amenable to exact treatment [1]. We showed that, for subthreshold input signals of sufficiently long periods, the phenomenon of SR can be accompanied by SR gains larger than unity. This is a genuine characterization of nonlinearity, as SR gains larger than unity are strictly forbidden within a linear response description [3, 4, 5]. The analytical results were corroborated by numerical simulations [2].

In the last few years, Gingl and collaborators [6, 7, 8] have carried out analog simulations of noisy bistable systems subject to driving forces of the type given in Eq. (1)

$$F(t) = \begin{cases} A & : \quad 0 \leq t < t_c \\ -A & : \quad \frac{T}{2} \leq t < \frac{T}{2} + t_c \\ 0 & : \quad \text{otherwise} \end{cases} \tag{1}$$

It is convenient to introduce the parameter $r = 2t_c/T$, measuring the fraction of a period during which this driving force has a nonvanishing value (the parameter r in the present paper corresponds exactly to what Gingl and coworkers term "duty cyle"). In [6, 7, 8], the SNR and the SR gain for subthreshold amplitude input signals with $r \leq 0.3$ were studied. These authors find SR gains larger than unity, and, also, that increasing the r value lowers the SNR gain. They rationalize their observations by noting that the input SNR increases as r increases, while the output SNR is less sensitive to the value of r. The case studied by us in [1, 2] corresponds to the largest possible value of the parameter r, namely, $r = 1$. It seems therefore of interest to extend our analysis to input signals with $r < 1$ in order to compare with the predictions of Gingl et al.

Let us consider a system characterized by a single degree of freedom, x, whose dynamics (in dimensionless units) is governed by the Langevin equation

$$\dot{x}(t) = -U'\left[x(t), t\right] + \xi(t), \tag{2}$$

where $\xi(t)$ is a Gaussian white noise of zero mean with $\langle \xi(t)\xi(s) \rangle = 2D\delta(t-s)$, and $-U'(x,t)$

represents the force stemming from the time-dependent, archetype bistable potential

$$U(x,t) = \frac{x^4}{4} - \frac{x^2}{2} - F(t)x, \tag{3}$$

with $F(t)$ given by Eq. (1). The one-time correlation function is defined as

$$C(\tau) = \frac{1}{T}\int_0^T \mathrm{d}t\, \langle x(t+\tau)x(t)\rangle_\infty. \tag{4}$$

It can be written exactly as the sum of two contributions: a coherent part, $C_{coh}(\tau)$, which is periodic in τ with period T, and an incoherent part, $C_{incoh}(\tau)$, which decays to 0 for large values of τ and reflects the correlation of the output fluctuations about its average (the noisy part of the output). The coherent part $C_{coh}(\tau)$ is given by [9, 10]

$$C_{coh}(\tau) = \frac{1}{T}\int_0^T \mathrm{d}t\, \langle x(t+\tau)\rangle_\infty \langle x(t)\rangle_\infty, \tag{5}$$

and $C_{incoh}(\tau)$ is obtained from the difference of Eq. (4) and Eq. (5). In the expressions above, the susbcript indicates that the averages are to be evaluated in the limit $t \to \infty$.

According to McNamara and Wiesenfeld [11], the output SNR, R_{out}, is defined in terms of the Fourier transform of the coherent and incoherent parts of $C(\tau)$ as

$$R_{out} = \frac{\lim_{\epsilon\to 0^+} \int_{\Omega-\epsilon}^{\Omega+\epsilon} \mathrm{d}\omega\, \tilde{C}(\omega)}{\tilde{C}_{incoh}(\Omega)}, \tag{6}$$

where $\tilde{H}(\omega)$ denotes the Fourier cosine transform of $H(\tau)$, i.e., $\tilde{H}(\omega) = 2/\pi \int_0^\infty \mathrm{d}\tau\, H(\tau)\cos(\omega\tau)$. Note that this definition of the output SNR differs by a factor 2, stemming from the same contribution at $\omega = -\Omega$, from the definitions used in earlier works [9, 10]. The periodicity of the coherent part gives rise to delta peaks in the spectrum. Thus, the only contribution to the numerator in Eq. (6) stems from the coherent part of the correlation function. The output SNR can then be expressed as

$$R_{out} = \frac{Q_u}{Q_l}, \tag{7}$$

where

$$Q_u = \frac{2}{T}\int_0^T \mathrm{d}\tau\, C_{coh}(\tau)\, \cos(\Omega\tau), \tag{8}$$

and

$$Q_l = \frac{2}{\pi}\int_0^\infty \mathrm{d}\tau\, C_{incoh}(\tau)\, \cos(\Omega\tau). \tag{9}$$

A nonmonotonic behavior of the SNR with the strength of the noise is a signature of the phenomenon of SR.

The signal-to-noise ratio for an input signal $F(t) + \xi(t)$ is given by

$$R_{inp} = \frac{\frac{1}{2}(f_1^2 + g_1^2)}{\frac{2}{\pi}D}. \tag{10}$$

where

$$f_1 = \frac{2A}{\pi}\sin\Omega t_c \tag{11}$$

and

$$g_1 = \frac{2A}{\pi}(1 - \cos\Omega t_c) \tag{12}$$

It is then clear that for fixed values of A and D, R_{inp} increases with r as pointed out in Ref. [8]. Also, for given A and r, R_{inp} decreases as D increases.

The SR gain, G, is defined as the ratio of the SNR of the output over that of the input; namely,

$$G = \frac{R_{out}}{R_{inp}}. \tag{13}$$

SR gain values larger than 1 have been obtained in driven nondynamical systems [6], in stochastic resonators with static nonlinearities driven by square pulses [12], or in noisy bistable systems driven by superthreshold input sinusoidal frequencies [13]. The existence of SR gains with values larger than 1 indicates a truly nonlinear SR.

Although the two-state approximation introduced in Ref. [1] can, in principle, be extended to analyze systems driven by input signals with $r < 1$, the analytical expressions obtained are too cumbersome to be of practical value. Thus, in the present work, we rely on the numerical treatment of the Langevin equation, Eq. (2), following the procedure detailed in Ref. [3].

In Fig. 1, we depict the behavior of R_{out} with the noise strength D, for input signals of the type given in Eq. (1), with subthreshold amplitude $A = 0.35$, fundamental frequency $\Omega = 0.0024$ and $r = 0.1, 0.4, 0.7, 0.95, 0.98, 1$. For all values of r, the characteristic nonmonotonic behavior of the SNR with D is obtained. As r increases, the maximum value of R_{out} increases. Namely, the longer the potential remains asymmetric during each half-cycle, the larger the maximum height in the SNR is. Therefore, R_{out} is quite sensitive to the duration of the pulses within each half-cycle.

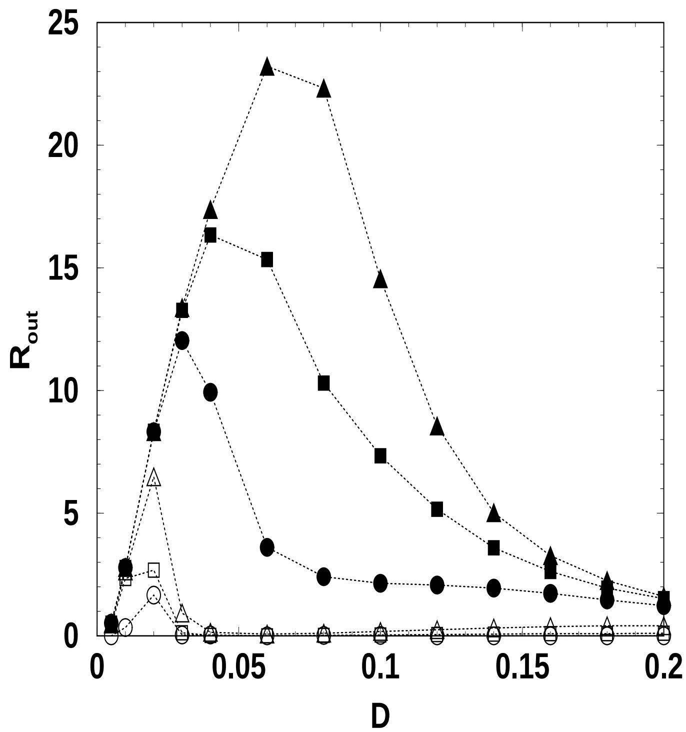

FIG. 1: The output signal-to-noise ratio, R_{out} *vs.* the noise strength, D, for $r = 0.1$ (circles), $r = 0.4$ (squares), $r = 0.7$ (triangles), $r = 0.95$ (filled circles), $r = 0.98$ (filled squares) and $r = 1.0$ (filled triangles). The input signal has a subthreshold amplitude $A = 0.35$ and a fundamental frequency $\Omega = 2\pi/T = 0.0024$.

In Fig. 2 the behavior of G with the noise strength D is depicted for the same values of r as in the previous figure. For all values of r considered, there exists a range of noise values such that G is larger than unity. The peak value of the SR gain has a nonmonotonic behavior with r. At the lowest value of r considered, the SR gain has a rather large peak value. Then, as r is increased, the peak of the SR gain decreases, in agreement with the observations in [7, 8]. As the duration of the pulses gets larger so that r is closer to 1, the tendency of the SR gain maximum reverses and a considerable increase in the maximum is observed.

The just mentioned features can be rationalized by noting that the SR gain depends on R_{out} and on R_{in}. As indicated above, for fixed values of the noise strength, D, and amplitude, A, the input R_{in} always increases with r. Also, for given values of A and r, R_{in} decreases monotonically with D. On the other hand, the results depicted in Fig. 1 indicate

that there are two main effects on the location of the maximum of R_{out} as r increases. First, as noted before, the maximum height increases as r increases. Second, the maxima appear at increasingly large values of D as the duration of the pulses increases. This second effect manifests itself clearly for pulses of sufficiently long duration, namely for r values larger than $r \approx 0.9$, while it is almost unnoticeable for smaller bvalues of r. For low values of r (let us say $r = 0.1$), even though the peak of R_{out} is the smallest one appearing in Fig. 1, the corresponding value of R_{in} is so small (due to the smallness of r) that the SR gain reaches the large values depicted in Fig. 2. As r increases, the height of the R_{out} maximum also increases, but appearing at an approximately constant value of the noise strength. Thus, the increase of R_{in} with r counterbalances the increase of R_{out} in such a way that the SR gain decreases. Finally, for long duration pulses, the shift to the right of the R_{out} maximum and the large increase in its height are the cause of the increase in the maximum gain observed in Fig. 2.

In conclusion, we observe that the behavior of the SR gain for pulses of relatively short duration is basically a consequence of the behavior of R_{in}, in agreement with the observation of Gingl et al. [6, 7, 8]. On the other hand, for r close to 1, the behavior of the SR gain is dominated by the output SNR.

Acknowledgments

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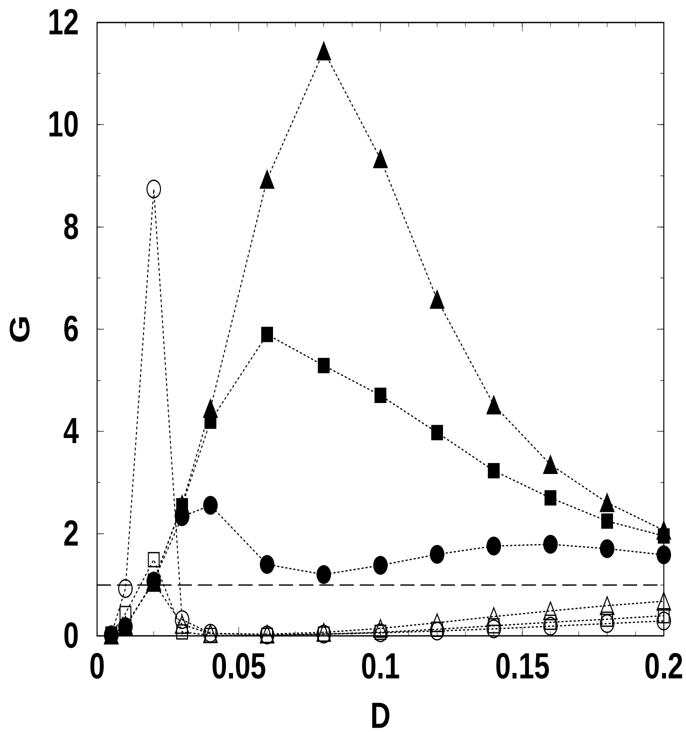

FIG. 2: The SR gain, G, *vs.* the noise strength, D, for the same parameter values as in Fig. (1).

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